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## FAST TRACK COMMUNICATION

# A decomposition of separable Werner states

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Online at [stacks.iop.org/JPhysA/40/F483](http://stacks.iop.org/JPhysA/40/F483)**Abstract**

We derive an integral convex combination of product states for a range of separable Werner states. Our method consists of expanding the sought-after local density operators in terms of Wigner operators. For dimension  $d = 2$ , our decomposition holds for the whole separable range of Werner states, while for  $d > 2$  it is valid for a subset of separable Werner states. We illustrate the general method with the explicit examples  $d = 2$  and  $d = 3$ .

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## 1. Introduction

Composite quantum systems are among the foci of quantum information theory. Bipartite quantum states, i.e. states of quantum systems that consist of two subsystems  $A$  and  $B$ , can be classified according to their property of being entangled or separable. In 1989, R Werner proposed a physically meaningful definition of separability of a quantum state [1], namely

$$\rho_{\text{sep}} = \sum_i p_i \rho_A^i \otimes \rho_B^i, \quad (1)$$

where  $p_i$  are probabilities, i.e.  $p_i \geq 0$  and  $\sum_i p_i = 1$ . In the integral version of this convex combination of product states, the discrete index  $i$  is replaced by some continuous variable  $\lambda$ , such that the integral product decomposition reads  $\rho_{\text{sep}} = \int d\lambda p(\lambda) \rho_A(\lambda) \otimes \rho_B(\lambda)$ , where  $p(\lambda) \geq 0$  and  $\int d\lambda p(\lambda) = 1$ . Given a separable state, its product decomposition is not unique and, in general, it is a hard task to find such a decomposition. If one does not know whether a given state is separable or entangled, it is difficult to prove whether a separable decomposition does or does not exist. This is the origin of the entanglement versus separability problem [2, 3].

In his seminal paper [1], Werner also introduced a certain family of states that is nowadays referred to as Werner states. Due to their symmetry properties, they play an important role in several contexts, e.g. in entanglement purification [4], entanglement properties for states

with white noise and the possible existence of bound entangled states with non-positive partial transpose [5].

For qubits, a separable decomposition of Werner states has been found in [6]. A different decomposition was recently derived in [7], again for two-dimensional subsystems. Here, we find a separable decomposition of Werner states in arbitrary dimensions. For dimension  $d = 2$ , our decomposition is different from both decompositions in [6] and [7].

## 2. Werner states

Werner states [1], in the following denoted as  $\rho_W$ , are a class of mixed states for bipartite quantum systems (where each of the two subsystems has dimension  $d$ ), which are invariant under the transformations  $U \otimes U$ , for any unitary  $U$ , i.e.  $\rho_W = (U \otimes U)\rho_W(U^\dagger \otimes U^\dagger)$ . The family of Werner states, characterized by one parameter  $f$ , is (in the original notation) given by the density operator

$$\rho_W = \frac{1}{d^3 - d}((d - f)\mathbb{1} + (df - 1)\mathbf{V}), \quad (2)$$

which acts on the  $d \times d$ -dimensional Hilbert space  $H_A \otimes H_B$  that is spanned by the state vectors of subsystems  $A$  and  $B$ . Here,  $\mathbb{1}$  is the identity operator on  $H_A \otimes H_B$ , and by  $\mathbf{V}$  we denote the swap operator, which acts as  $\mathbf{V}|\phi\rangle_A \otimes |\psi\rangle_B = |\psi\rangle_A \otimes |\phi\rangle_B$ . Positivity of  $\rho_W$  implies for the parameter  $f = \text{tr}(\rho_W \mathbf{V})$  that  $-1 \leq f \leq 1$ . The Werner state  $\rho_W$  is separable, i.e. classically correlated, iff  $0 \leq f \leq 1$ . In the following, we will regard the Hilbert space of a subsystem as the state space of a particle with spin  $j$ , where  $d = 2j + 1$ . The basis states are denoted by  $|jm\rangle$ , where  $m = j, j - 1, \dots, -j$ . It is known that  $\mathbf{V}$  has the following form in terms of the Wigner operators  $T_q^k$  [8]:

$$\mathbf{V} = \sum_{k=0}^{2j} \sum_{q=-k}^k (-1)^q T_q^k \otimes T_{-q}^k, \quad (3)$$

where  $T_q^k$  denotes the  $(2j + 1) \times (2j + 1)$  matrix with the elements given by [8]

$$\langle jm_1 | T_q^k | jm_2 \rangle = \left( \frac{2k + 1}{2j + 1} \right)^{1/2} C_{m_1 q m_2}^{j k j}. \quad (4)$$

The coefficients  $C_{m_1 q m_2}^{j k j}$  are Clebsch–Gordan coefficients. We use the notation  $|JMj_1 j_2\rangle = \sum_{m_1, m_2} C_{m_1 m_2 M}^{j_1 j_2 J} |j_1 m_1\rangle \otimes |j_2 m_2\rangle$ , where  $J$  is the total angular momentum,  $M$  is its third component,  $j_i$  is the angular momentum of particle  $i$  and  $m_i$  is its third component, with  $m_1 + m_2 = M$ . Note that the  $(2j + 1)^2$  operators  $T_q^k$  generate the group  $U(2j + 1)$ . The matrices  $T_q^k$  obey the orthogonality relations

$$\text{tr}(T_{q_1}^{k_1} \cdot (T_{q_2}^{k_2})^\dagger) = \delta_{k_1 k_2} \delta_{q_1 q_2} \quad (5)$$

and are traceless for  $k \neq 0$  [8]. For  $k = 0$ , we have

$$T_0^0 = (2j + 1)^{-1/2} \mathbb{1}. \quad (6)$$

Using equation (3), we can rewrite the Werner state (2) in terms of the Wigner operators:

$$\rho_W = \frac{1}{d^3 - d} \left( (d - f)\mathbb{1} + (df - 1) \sum_{k=0}^{2j} \sum_{q=-k}^k (-1)^q T_q^k \otimes T_{-q}^k \right). \quad (7)$$

### 3. Decomposition of separable Werner states

In general, a separable state can be written in the integral decomposition [1]

$$\rho_{\text{sep}} = \int d\lambda p(\lambda) [\rho_A(\lambda) \otimes \rho_B(\lambda)], \quad (8)$$

where

$$p(\lambda) \geq 0, \quad \int d\lambda p(\lambda) = 1,$$

and  $\rho_A(\lambda) \geq 0$ ,  $\rho_B(\lambda) \geq 0$  represent density operators of subsystems  $A$  and  $B$ , respectively. Here,  $\lambda$  has to be understood as a symbol for one or more continuous variables.

We can express an arbitrary density operator  $\rho(\lambda)$  of a spin- $j$  particle via the operator decomposition

$$\rho(\lambda) = \sum_{k=0}^{2j} \sum_{q=-k}^k y_{kq}(\lambda) T_q^k,$$

where the expansion coefficients  $y_{kq}$  are given by

$$y_{kq}(\lambda) = \text{tr}(\rho(\lambda) T_q^{k\dagger}), \quad (9)$$

due to the orthogonality relation (5).

Using this decomposition for both  $\rho_A(\lambda)$  and  $\rho_B(\lambda)$ , we can rewrite a separable Werner state (7) as

$$\rho_W = \int d\lambda p(\lambda) \left[ \sum_{k_1=0}^{2j} \sum_{k_2=0}^{2j} \sum_{q_1=-k_1}^{k_1} \sum_{q_2=-k_2}^{k_2} y_{k_1 q_1}(\lambda) y_{k_2 q_2}(\lambda) T_{q_1}^{k_1} \otimes T_{q_2}^{k_2} \right]. \quad (10)$$

A comparison of (7) with (10) leads to the following condition on  $y_{k_1 q_1}(\lambda)$  and  $y_{k_2 q_2}(\lambda)$ :

$$\int d\lambda p(\lambda) y_{k_1 q_1}(\lambda) y_{k_2 q_2}(\lambda) \sim \delta_{k_1, k_2} \delta_{q_1, -q_2}. \quad (11)$$

We can conclude that  $y_{k_i q_i}(\lambda)$  with  $i = 1, 2$  are orthogonal, with a weight function  $p(\lambda)$ . Remember that the Werner state on the left-hand side of equation (10) depends on the parameter  $f$ , which is not explicitly written here. Thus, the expansion coefficients  $y_{k_1 q_1}(\lambda)$  and  $y_{k_2 q_2}(\lambda)$  and the weight function  $p(\lambda)$  will in general be functions of  $f$  as well.

It is possible to satisfy the orthogonality condition in equation (11) in the following way: let us consider  $\lambda$  as a symbol for two parameters. By defining  $\lambda = \{\theta, \varphi\}$  with  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi \leq 2\pi$  and  $p(\theta, \varphi) = 1/4\pi$  [7], relation (11) takes the form

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin \theta y_{k_1 q_1}(\theta, \varphi) y_{k_2 q_2}(\theta, \varphi) \sim \delta_{k_1, k_2} \delta_{q_1, -q_2}. \quad (12)$$

Thus, we can use for  $y_{kq}(\theta, \varphi)$  the spherical harmonics  $Y_q^k(\theta, \varphi)$ . One can readily check that the decomposition (10) can be rewritten as

$$\rho_W = \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin \theta [\rho_A(\theta, \varphi) \otimes \rho_B(\theta, \varphi)], \quad (13)$$

where

$$\rho_A(\theta, \varphi) = \frac{1}{2j+1} \mathbb{1} + \frac{1}{(2j+1)} \frac{(2j+1)f-1}{(2j+1)^2-1} \sum_{k=1}^{2j} \sum_{q=-k}^k \eta_{kq}^{-1} Y_q^k(\theta, \varphi) T_q^k, \quad (14)$$

$$\rho_B(\theta, \varphi) = \frac{1}{2j+1} \mathbb{1} + \sum_{k=1}^{2j} \sum_{q=-k}^k \eta_{kq} Y_q^k(\theta, \varphi) T_q^k, \quad (15)$$

with  $\eta_{kq}$  being arbitrary parameters. We note that hermiticity of  $\rho_{A,B}(\theta, \varphi)$  implies that  $\eta_{kq}^* = \eta_{k, -q}$ . We have used the properties of the spherical harmonics  $Y_q^k(\theta, \varphi)$ , namely equation (12) and  $Y_q^{k*}(\theta, \varphi) = (-1)^q Y_{-q}^k(\theta, \varphi)$ . The above combination of product states for  $\rho_W$  still contains the freedom in choosing the parameters  $\eta_{kq}$ . Note, however, that at this point we have not yet shown positivity of the local operators, and the notation  $\rho_A$  and  $\rho_B$  is merely suggestive. The reader will have noted that the prefactors of the sums in expressions (14) and (15) are not identical. We have chosen this asymmetric decomposition on purpose, as will be explained below. The decomposition (13) could have equally well been formulated in a symmetric way by defining

$$\rho_A^{(s)}(\theta, \varphi) = \frac{1}{2j+1} \mathbb{1} + \sqrt{\frac{1}{(2j+1)} \frac{(2j+1)f-1}{(2j+1)^2-1}} \sum_{k=1}^{2j} \sum_{q=-k}^k \eta'_{kq} Y_q^k(\theta, \varphi) T_q^k, \quad (16)$$

$$\rho_B^{(s)}(\theta, \varphi) = \frac{1}{2j+1} \mathbb{1} + \sqrt{\frac{1}{(2j+1)} \frac{(2j+1)f-1}{(2j+1)^2-1}} \sum_{k=1}^{2j} \sum_{q=-k}^k \eta'_{kq} Y_q^k(\theta, \varphi) T_q^k. \quad (17)$$

In this version,  $\rho_A^{(s)}(\theta, \varphi)$  and  $\rho_B^{(s)}(\theta, \varphi)$  have a nice symmetric form; however, their positivity conditions are quite difficult to analyse. It is obvious that the separability property of the Werner state does not depend on the form of the local density operators; therefore, we choose to use the local operators (14) and (15) because they allow us to determine the positivity constraints in an easier way.

If all eigenvalues of the local operators (14) and (15) are positive, we found a valid separable decomposition of the Werner state  $\rho_W$  in equation (13). The calculation of the eigenvalues of the local density operators for  $j \geq 1$  is a difficult task. We now simplify the analysis by choosing  $\eta_{kq} = \eta_k$ . Since

$$Y_q^k(\theta, \varphi) = \sqrt{\frac{2k+1}{4\pi}} D_{q0}^{k*}(\theta, \varphi), \quad (18)$$

where  $D_{qm}^k(\theta, \varphi)$  is the Wigner rotation matrix, defined via the transformation of  $T_q^k$  as a tensor [8], i.e.

$$U : T_m^k \rightarrow UT_m^kU^\dagger = \sum_{q=-k}^k D_{qm}^k(\theta, \varphi) T_q^k, \quad (19)$$

inserting equation (18) into equations (14) and (15), using equation (19) and the fact that  $D_{qm}^{k*}(\theta, \varphi) = D_{qm}^k(\theta, -\varphi)$ , we can rewrite the local density matrices in the form

$$\rho_A(\theta, \varphi) = U_A(\theta, -\varphi) \left( \frac{1}{2j+1} \mathbb{1} + \frac{1}{(2j+1)} \frac{(2j+1)f-1}{(2j+1)^2-1} \sum_{k=1}^{2j} \eta_k^{-1} \sqrt{2k+1} T_0^k \right) U_A^\dagger(\theta, -\varphi), \quad (20)$$

$$\rho_B(\theta, \varphi) = U_B(\theta, -\varphi) \left( \frac{1}{2j+1} \mathbb{1} + \sum_{k=1}^{2j} \eta_k \sqrt{2k+1} T_0^k \right) U_B^\dagger(\theta, -\varphi). \quad (21)$$

Here,  $U(\theta, \varphi)$  denotes the unitary irreducible representation of the  $SO(3)$  group on the state space spanned by  $|jm\rangle$ , with  $m = j, j-1, \dots, -j$ , and is defined as

$$U(\theta, \varphi) = \exp(-i\varphi J_z) \exp(-i\theta J_y).$$

Note that in the unitary operator we have already omitted the third Euler angle  $\gamma$ , by dropping the factor  $\exp(i\gamma J_z)$ , because we are using the eigenbasis of  $J_z$ .

Thus, we can rewrite the Werner state (2) in the form

$$\rho_W = \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta [U_A(\theta, -\varphi) \otimes U_B(\theta, -\varphi)] \times [\rho_A \otimes \rho_B][U_A(\theta, -\varphi)^\dagger \otimes U_B(\theta, -\varphi)^\dagger], \tag{22}$$

where

$$\rho_A = \frac{1}{2j+1} \mathbb{1} + \frac{1}{(2j+1)} \frac{(2j+1)f-1}{(2j+1)^2-1} \sum_{k=1}^{2j} \eta_k^{-1} \sqrt{2k+1} T_0^k, \tag{23}$$

$$\rho_B = \frac{1}{2j+1} \mathbb{1} + \sum_{k=1}^{2j} \eta_k \sqrt{2k+1} T_0^k. \tag{24}$$

It is easy to see that  $\rho_A$  and  $\rho_B$  are diagonal matrices. To show this one uses (4) and the properties of the Clebsch–Gordan coefficients, namely  $C_{m_1 q m_2}^{j k j} = 0$  if  $m_1 + q \neq m_2$ . Thus the diagonal elements, i.e. eigenvalues of  $\rho_A$  and  $\rho_B$ , are given by

$$\lambda_m(\rho_A) = \frac{1}{2j+1} + \frac{1}{(2j+1)^{3/2}} \frac{(2j+1)f-1}{(2j+1)^2-1} \sum_{k=1}^{2j} \eta_k^{-1} (2k+1) C_{m 0 m}^{j k j} \tag{25}$$

and

$$\lambda_m(\rho_B) = \frac{1}{2j+1} + \frac{1}{\sqrt{2j+1}} \sum_{k=1}^{2j} \eta_k (2k+1) C_{m 0 m}^{j k j}, \quad m = j, j-1, \dots, -j. \tag{26}$$

If all  $\lambda_m(\rho_A)$  and  $\lambda_m(\rho_B)$  are positive, we found a separable form of the Werner state (2). We note that positivity of  $\lambda_m(\rho_A)$  and  $\lambda_m(\rho_B)$  for all  $m$  implies that  $\sum_{m=-j}^j \lambda_m(\rho_A) \lambda_m(\rho_B) \geq 0$ . Owing to the fact that [8]

$$\sum_{m=-j}^j C_{m 0 m}^{j k_1 j} C_{m 0 m}^{j k_2 j} = \frac{2j+1}{2k+1} \delta_{k_1 k_2} \quad \text{and} \quad \sum_{m=-j}^j C_{m q m}^{j k j} = 0 \quad \text{for } k \neq 0, \tag{27}$$

one finds  $\sum_{m=-j}^j \lambda_m(\rho_A) \lambda_m(\rho_B) = f$ . Thus,  $f \geq 0$  is a necessary condition for positivity of  $\rho_A$  and  $\rho_B$ . It remains to show the sufficient conditions for positivity of  $\rho_A$  and  $\rho_B$ . At this point it becomes clear why the asymmetric form, chosen above for  $\rho_A$  and  $\rho_B$ , is advantageous: we can determine the free parameters  $\eta_k$  from the positivity condition for  $\rho_B$  and then find the range of  $f$  for which  $\rho_A$  is positive.

We note that for the simplest case of spin  $j = 1/2$  the present decomposition is, due to the isomorphism between  $SO(3)$  and  $SU(2)$ , equivalent to

$$\rho_W = \int dU [U \otimes U] \cdot [\rho_A \otimes \rho_B] \cdot [U \otimes U]^\dagger, \tag{28}$$

where the integral is extended to all unitary operators acting on the two-dimensional Hilbert space, with  $\int dU = 1$  and  $dU$  representing the standard Haar measure on the group  $SU(2)$ . Hence, we expect that the inequalities  $\lambda_{\pm 1/2}(\rho_{A,B}) \geq 0$  will yield  $0 \leq f \leq 1$ . For higher dimensions, however, there is no such simple argument, and it turns out that for higher spins the decomposition (22) is a separable decomposition, i.e.  $\rho_{A,B} \geq 0$ , only in the range  $0 \leq f \leq f_0(j)$ , where  $f_0(j) < 1$ .

The presented method can also be used to find a separable form for more general states, e.g., states which are invariant under product representations of the group  $SO(3)$  of three-dimensional rotations; see [9].

#### 4. Examples

Let us consider explicitly the two lowest dimensions, namely  $d = 2$  (spin  $j = 1/2$ ) and  $d = 3$  (spin  $j = 1$ ).

##### 4.1. Case $j = 1/2$

Here, the index  $k$  in equations (25) and (26) takes one value,  $k = 1$ , and we have one free parameter  $\eta_1$ . Thus, for qubits we have the following inequalities:

$$\lambda_{1/2}(\rho_B) = \frac{1}{2}(1 + \sqrt{6}\eta_1) \geq 0, \quad \lambda_{-1/2}(\rho_B) = \frac{1}{2}(1 - \sqrt{6}\eta_1) \geq 0.$$

This leads to the condition

$$|\eta_1| \leq \frac{1}{\sqrt{6}}. \quad (29)$$

Positivity of  $\lambda_{\pm 1/2}(\rho_A)$  translates to the constraints

$$\lambda_{1/2}(\rho_A) = \frac{1}{2} + \frac{\sqrt{6}2f - 1}{12\eta_1} \geq 0, \quad (30)$$

$$\lambda_{-1/2}(\rho_A) = \frac{1}{2} - \frac{\sqrt{6}2f - 1}{12\eta_1} \geq 0. \quad (31)$$

The solution of the inequalities (29)–(31) is

$$|\eta_1| \leq \frac{1}{\sqrt{6}} \quad \text{and} \quad 0 \leq f \leq 1.$$

Thus, as mentioned above, for  $j = 1/2$  our decomposition is valid for the whole separable range of the Werner state family.

##### 4.2. Case $j = 1$

For  $j = 1$ , we have  $k = 1, 2$ . We can use two free parameters  $\eta_1, \eta_2$ . Positivity of  $\lambda_m(\rho_B)$  for  $m = -1, 0, +1$  implies the following inequalities:

$$\begin{aligned} \lambda_1(\rho_B) &= \frac{1}{6}(2 + 3\sqrt{6}\eta_1 + \sqrt{30}\eta_2) \geq 0, \\ \lambda_{-1}(\rho_B) &= \frac{1}{6}(2 - 3\sqrt{6}\eta_1 + \sqrt{30}\eta_2) \geq 0, \\ \lambda_0(\rho_B) &= \frac{1}{3}(1 - \sqrt{30}\eta_2) \geq 0. \end{aligned} \quad (32)$$

The eigenvalues of  $\lambda_m(\rho_A)$  read

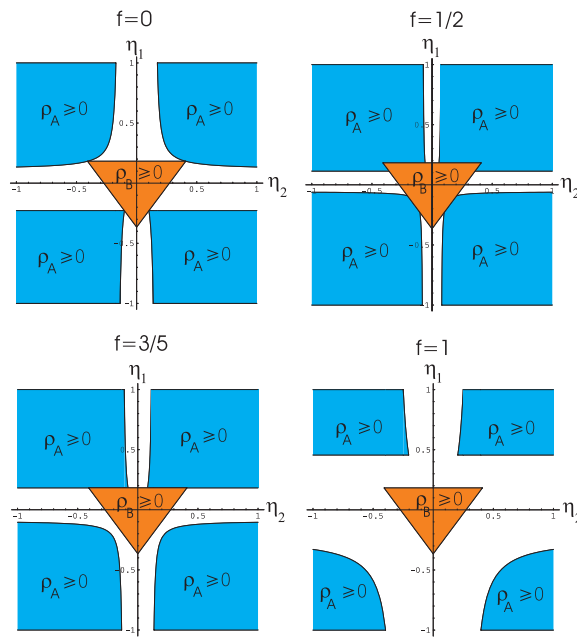
$$\begin{aligned} \lambda_1(\rho_A) &= \frac{1}{144} \left( \frac{\eta_1(\sqrt{30}(3f - 1) + 48\eta_2) + 3\sqrt{6}(3f - 1)\eta_2}{\eta_2\eta_1} \right), \\ \lambda_{-1}(\rho_A) &= \frac{1}{144} \left( \frac{\eta_1(\sqrt{30}(3f - 1) + 48\eta_2) - 3\sqrt{6}(3f - 1)\eta_2}{\eta_2\eta_1} \right), \\ \lambda_0(\rho_A) &= \frac{1}{72} \left( 24 - (3f - 1) \frac{\sqrt{30}}{\eta_2} \right). \end{aligned} \quad (33)$$

After a lengthy calculation, we conclude that positivity of  $\rho_A$  holds for

$$0 \leq f \leq \frac{3}{5},$$

i.e. for the case  $j = 1$ , our decomposition is valid only within a certain range of the parameter  $f$  and not for the whole separable interval  $0 \leq f \leq 1$ . Note that our separable decomposition holds for Werner states that are ‘close’ to entangled states.

Figure 1 shows the solutions of inequalities  $\lambda_m(\rho_{A,B}) \geq 0$  for different values of  $f$ , for spin  $j = 1$ . One can see that the common area of filled regions vanishes for  $f > \frac{3}{5}$ .



**Figure 1.** Case spin  $j = 1$ : Solutions of inequalities (32) and (33) (filled regions) for different values of  $f = 0, \frac{1}{2}, \frac{3}{5}$  and 1.

(This figure is in colour only in the electronic version)

## 5. Conclusions

In summary, we have used an angular momentum approach to find in principle a separable decomposition of Werner states in any dimension. Our main idea is to express the local density operators and the global operator in terms of Wigner operators. The eigenvalues of the local operators are found to be certain combinations of Clebsch–Gordan coefficients. Some free parameters allow us to guarantee the positivity of the local density operators. For dimension  $d = 2$  our decomposition holds for all separable Werner states, while for higher dimensions it is only valid for a certain range of separable Werner states. The size of this range depends on the dimension. We verified that our decomposition is valid for  $f = 0$  up to dimension  $d = 5$ . It is still an open task to find a product decomposition of *all* separable Werner states in any dimension.

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